Recent Results in the Mathematical Modeling of Financial Bubbles

Philip Protter, Columbia University

Conference in Honor of Professor Vlad BALLY

Parts of this talk are based on joint work with **Aditi Dandapani**, others with **Shihao Yang**, and still other parts with **Jean Jacod**

October 7, 2015

How do we model financial bubbles?

- We are given a filtered complete probability space: $(\Omega, \mathcal{F}, P, \mathbb{F})$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions, and contains at least one Brownian motion
- We let S denote our nonnegative stock price process, & assume interest rates are zero
- Let $\mathbb{Q}$ denote all risk neutral measures Q
- The **Fundamental Price** of a stock, denoted $S^* = (S^*_t)_{0 \leq t \leq T}$, is the conditional expectation

$$S^*_t = E_Q\{ \textbf{ All cash flows after time } t | \mathcal{F}_t\} \qquad (1)$$

- It is impossible really to know S^*_t

Mathematics to the Rescue

- NFLVR$\Rightarrow S_t \geq S_t^*$ a.s.
- $\beta_t = S_t - S_t^* \geq 0$ is the bubble process
- **Theorem[Jarrow, P^2, Shimbo] 2010:** On a compact time interval $[0, T]$ a stock price is undergoing bubble pricing if and only if the bubble process $\beta_t > 0$ and $\beta_t = S_t - S_t^*$ **is a strict local martingale under** $Q \in \mathbb{Q}$
- This theorem builds on work of **Lowenstein & Willard**, and **Cox & Hobson**
- There is a lot of subsequent work by **E. Bayraktar, F. Biagini, H. Föllmer, C. Kardaras, A. Nikeghbali, A. Roch, M. Schweitzer,** and others

Formation of Bubbles

- We put ourselves in the framework of an incomplete market, so that there is an infinite number of risk neutral measures
- We model the risky price process using a stochastic volatility paradigm:

$$dS_t = \sigma(S_t, \nu_t)dB_t + b(S_t, \nu_t)dt \tag{2}$$

- Jarrow, P^2, and Shimbo originally proposed regime changes occurring at stopping times $T_1 \leq T_2 \leq \dots$, where the coefficients, or especially the risk neutral measure, changes at each time T_i
- This was improved in the work of **F. Biagini, H. Föllmer, and S. Nedelcu** where they show that a continuous change of risk neutral measures can let the price process evolve from a martingale into a strict local martingale, thereby modeling the birth of a bubble

- Indeed, to make (2) more explicit, let it be of the form

$$\begin{aligned} dS_t &= \sigma(S_t, \nu_t)dB_t + \mu(S_t, \nu_t)dt \\ d\nu_t &= f(\nu_t)dW_t + g(\nu_t)dt \end{aligned} \qquad (3)$$

where $d[B, W]_t = \rho dt$
- The idea of **BFN** is that as the risk neutral measures change, the drift of ν in (3) changes in distribution in such a way as to render S a strict local martingale
- This is quite elegant but does not directly connect bubble birth to economic reasoning
- Finally, they use the classic and renowned 1998 results of Carlos Sin, where he gives necessary and sufficient conditions for a solution of an SDE of a specific form to be a strict local martingale.

A Possible Cause for Bubbles

- What causes a stock to enter into bubble pricing (ie, speculative pricing)?
- This is the subject of many papers by economists, such as **José Scheinkman & Harrison Hong**
- In work with **Aditi Dandapani**, a PhD student at Columbia, we add information to the filtration and find that doing so can lead to bubbles
- We do this using an initial expansion technique developed by Jean Jacod in the 1980s
- More precisely, we use the work of **PL Lions and M Musiela** on characterizing when solutions of specific types of SDEs are strict local martingales, which they did without regard to the theory of bubbles
- The Lions-Musiela results are close to those of Sin, but are more general

- Motivated by the possibility of incorrect pricing in financial markets, Lions & Musiela studied Heston type models of the form

$$\begin{aligned} dS_t &= S_t \nu_t dB_t \\ d\nu_t &= f(\nu_t) dW_t + b(\nu_t) dt \end{aligned} \tag{4}$$

where again $d[B,W]_t = \rho dt$

- We work on a time interval $[0, T]$ (compact)
- **Theorem (Lions & Musiela)**

If $\limsup_{x\to\infty} \frac{\rho x f(x) + b(x)}{x} < \infty$ then S is a nonnegative martingale

$$\text{If } \liminf_{x\to\infty} \frac{\rho x f(x) + b(x)}{\phi(x)} > 0,$$

where $\phi(x)$ is increasing, positive, smooth, and $\int_a^\infty \frac{1}{\phi(x)} ds < \infty$, then S is a strict local martingale.

- We add a countable partition of events, at a time t_0 to the underlying filtration $\mathbb{F}$, to get a larger filtration $\mathbb{G}$
- This changes the semimartingale decompositions in (4) using $(\mathbb{F}, P)$ and we have to remove an extra drift
- We then do a Girsanov transformation to calculate the new risk neutral measures by removing the extra drift, and we choose one we call Q, and show that under the right hypotheses (eg, $\rho > 0$, and with the correct assumptions on f and b in (16)) we get that S changes from a martingale under $(\mathbb{F}, P)$ to a strict local martingale under $(\mathbb{G}, Q)$

- My thesis student at Columbia **Aditi Dandapani** has proved the following
- **Theorem:** Let B and W be two Brownian motions, and let S and ν satisfy

$$\begin{aligned} dS_t &= S_t \nu_t dB_t \\ d\nu_t &= f(\nu_t)dW_t + b(\nu_t)dt \end{aligned} \tag{5}$$

where $d[B,W]_t = \rho dt$. Suppose that f and b are such that

$$\limsup_{x\to\infty} \frac{\rho x f(x) + b(x)}{x} < \infty, \text{ and}$$

$$\liminf_{x\to\infty} \frac{(\rho x f(x) + b(x) + \varepsilon f^2(x) + \varepsilon(\rho+1)f(x))}{\phi(x)} > 0$$

Then S in (5) is a $(P, \mathbb{F})$ martingale, and a $(Q, \mathbb{G})$ strict local martingale.

- An example of a choice of coefficients that works is $f(x) = x$ and $b(x) = x - \rho x^2$
- A key tool used in the proofs of Carlos Sin, Lions-Musiela, and also Aditi, is Feller's test for explosions in the equation for ν
- We also use a relatively new concept of **locally having no arbitrage**
- This result can be extended to slightly more general frameworks, with more general coefficients

How Life Gets Messy when Data is Involved

- We want to be able to detect, from data, when pricing is in a bubble
- To begin, we choose a quite specific model
- We assume our stock price S solves an SDE of the form

$$dS_t = \sigma(S_t)dB_t + b(S_t, \nu_t)dt; \quad S_0 = 1 \tag{6}$$

- This is an incomplete market setting
- With this framework, under any risk neutral measure Q equivalent to P, we always get the same equation:

$$dS_t = \sigma(S_t)dB_t$$

and this is key

- It is not realistic perhaps, since we do not have ν in the volatility, but it might be accurate for short amounts of time
- For equation (6) we have techniques to estimate $\sigma(x)$ developed by **Florens-Zmirou**, and **Jacod (2000)**. We (**R. Jarrow, Y. Kchia**, and $\mathbf{P}^2$) use a similar technique
- The (non parametric) estimate for σ has two problems:
  [a] The estimate is noisy
  [b] We can only estimate $x \mapsto \sigma(x)$ for x in the range of S_t, for $0 \leq t \leq T$

- Under $Q \in \mathbb{Q}$ we have that (2) becomes

$$dS_t = \sigma(S_t)dB_t; \quad S_0 = 1 \tag{7}$$

- We can use a Theorem of **Delbaen and Shirakawa (2002):**
  Theorem[D& S, 2002]: The process S in (7) is a nonnegative strict local martingale if and only if

$$\begin{aligned} \text{[a]} \int_0^{\varepsilon} \frac{x}{\sigma(x)^2} dx &= \infty, \text{ and} \\ \text{[b]} \int_{\varepsilon}^{\infty} \frac{x}{\sigma(x)^2} dx &< \infty \end{aligned}$$

- This theorem was improved later by **Kotani** and **Mijatovic & Urusov**
- **Since S^* is always a martingale, $\beta = S - S^*$ is a strict local martingale (and hence a bubble) if and only if S is a strict local martingale,** which means if and only if we have (a) & (b) above

Interpolation & Extrapolation

- We need to smooth σ to get a function to check the Delbaen-Shirakawa conditions
- We use a Reproducing Kernel Hilbert Space (**RKHS**) technique to smooth our estimate of σ
- The RKHS technique smooths the graph of $x \mapsto \sigma(x)$ in a way analogous to using least squares to fit a line to a cloud of points; this time we fit a curve
- But this is not enough: To check (a) & (b) we need to know the behavior of $x \mapsto \sigma(x)$ asymptotically as $x \to \infty$
- **Now for the part that is "louche:"** we extrapolate $x \mapsto \sigma(x)$ to all of $[0, \infty)$ using again an RKHS technique and an optimization criterion

Data Enters

- We got tick data for a 13 year period (2000-2013) from the Wharton Data Research Service (WRDS)
- Tick data is too noisy, so we use an idea of **L. Zhang, P. Myland, and Y. Aït-Sahalia (2005)** to perform a subsampling that reduces the noise
- We look for when σ behaves such that the integral

$$\int_{\varepsilon}^{\infty} \frac{x}{\sigma(x)^2} dx < \infty$$

which means we have that S is a strict local martingale and therefore $\beta > 0$, and we have a bubble

- We get a lot of false readings, and instability of the test, so we smooth the results using a Hidden Markov Model technique (HMM)
- We get a large number of fleeting bubble readings, so we impose a 5% filter: The stock price must rise more than 5% to signify the birth of a bubble, and it must later fall 5% to signify the death of a bubble, given that the test reads positive for a bubble
- The imposition of the 5% filter distorts a bit the results, and they should be interpreted with that in mind
- Using this technique, **we can compute the empirical distribution of the lifetimes of financial bubbles**

The Results

We get a histogram of the results which is well fit by a **generalized gamma distribution**

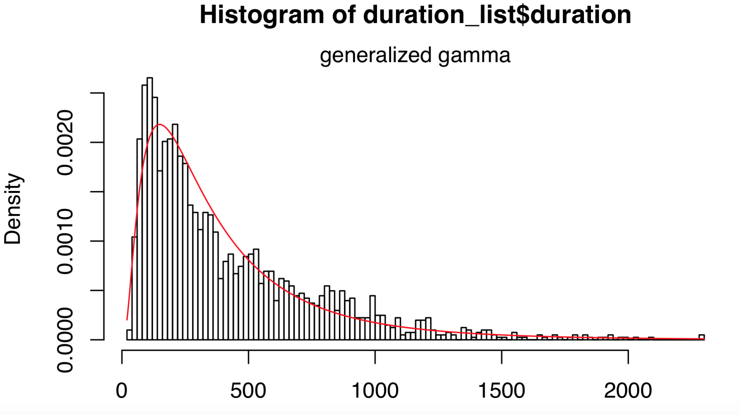

Figure: Histogram of bubble lifetimes

- We use MLE estimators to discern the parameters
- Next we try goodness of fit tests, using a QQ plot and a Kolmogorov-Smirnoff test

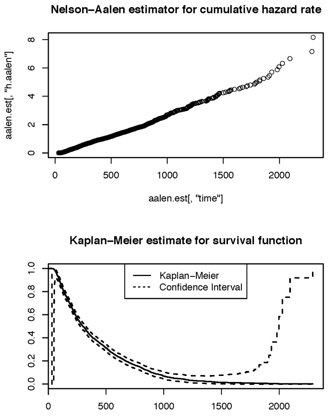

Figure: Goodness of Fit Graphs

- The generalized gamma has the density

$$f_G(t) = \frac{\lambda p(\lambda t)^{p\kappa-1} e^{-(\lambda t)^p}}{\Gamma(\kappa)} \tag{8}$$

  and p and κ are shape parameters.
- If $\kappa = 1$ then the Generalized Gamma reduces to the Weibull. It also extends the log normal, the exponential, and of course the gamma distribution.
- We performed an MLE estimate for the parameters, given our extensive data set. We obtained
  - (a.) The MLE estimate for $p = 0.1291201$
  - (b.) The MLE estimate for $\lambda = (1/3.655156)e^{-13}$
  - (c.) The MLE estimate for $\kappa = 84.74055$

Why do we get the generalized gamma distribution?

- The generalized gamma is a bit esoteric as a distribution, unless you work in survival analysis, and are interested in the distribution of lifetimes
- In 1967, JH Lienhard and PL Meyer proposed a derivation of the generalized gamma distribution from the standpoint of problems in physics
- We can mimic their derivation, adjusting it to fit the situation of financial bubbles
- We take the convention that all bubbles begin at time $t = 0$. We can achieve this by simply translating the bubble birth time to $t = 0$.

- We uniformly partition $\mathbb{R}_+$ into intervals $[t_{i-1}, t_i)$ of length Δt.
- Next we let N_i denote the number of bubbles still alive in $[t_{i-1}, t_i)$.
- Let N be the total number of bubbles in our universe. Then

$$\frac{N_i}{N} \text{ is the proportion of bubbles still alive at time } t_{i-1} \tag{9}$$

- We assume that the proportion of bubbles alive decreases geometrically with time, and we express this as

$$\sum_{i=1}^{\infty} \left(\frac{N_i}{N} \right) t_i^{\beta} = K \text{ for constants } \beta > 0, K > 0 \tag{10}$$

- We also assume the death rate of bubbles alive at time t_{i-1} is proportional to a power of t_i.
- This gives that the likelihood of bubble death increases geometrically with age.
- Thus if we let g_i denote the number of bubble deaths in $[t_{i-1}, t_i)$, we assume

$$g_i = A t_i^{\alpha-1} \tag{11}$$

so that g_i is proportional to a power of t, and the proportionality constant is A.

- We next look for the most probable distribution satisfying (9),(10) and (11).
- This gets complicated, and here we present only a sketch of the ideas.
- Let W be the number of ways bubbles can die in $[t_{i-1}, t_i)$ given that they are alive at time t_{i-1}, for all intervals $[t_{i-1}, t_i)$ over $[0, \infty)$.
- For example, bubbles can have a dramatic death, or they can die slowly, with a whimper, and one can give descriptions in between.
- There can also be varying economic explanations for why bubbles die, such as disagreements among different agents as to the state of current conditions; see for example the classic paper of **J. Scheinkman and W. Xiong**
- We obtain the following:

$$W = N! \prod_{i=1}^{\infty} \frac{g_i^{N_i}}{N_i!} \tag{12}$$

- We let $\tilde{N}_i$ denote the values of N_i that maximize W.
- One can then show

$$\frac{\tilde{N}_i}{N} = \frac{\Delta t[\beta \left(\frac{\beta K}{\alpha}\right)^{-\alpha/\beta}]}{\Gamma\left(\frac{\alpha}{\beta}\right)} t_i^{\alpha-1} \exp\left(-\frac{\alpha}{\beta}\frac{t_i^\beta}{K}\right) \tag{13}$$

- The idea for showing (13) is to maximize $\log(W)$ and to use that the maximum occurs when

$$d\log(W) = \sum_{i=1}^{\infty} [\log(At_i^{\alpha-1} - \log(N_i)]dN_i = 0$$

and then to use Stirling's approximation for factorial

- Our last step is to use this discrete distribution which we have found to approximate the continuous distribution
- Let τ be a stopping time. The probability that a given bubble is still alive in the interval $[t_{i-1}, t_i)$ is given by $P(t_{i-1} \leq \tau < t_i) = \tilde{N}_i / N$.
- Let the sought density f satisfy

$$\frac{\tilde{N}_i}{N} = \int_{t_{i-1}}^{t_i} f(s) ds = \Delta t f(\xi)$$

by the mean value theorem, for some ξ such that $t_{i-1} \leq \xi \leq t_i$

- Next let $\Delta t \to 0$ and use (13) to get

$$f(t) = \left[\frac{\beta}{\Gamma(\alpha/\beta)} \left(\frac{\alpha}{\beta K} \right)^{\alpha/\beta} \right] t^{\alpha-1} \exp\left(-\frac{\alpha}{\beta} \frac{t^\beta}{K} \right), \text{ for } t \geq 0 \tag{14}$$

where of course α, β and k are all positive (so that $f \geq 0$)
- Finally, if we make the change of variable $a = (\beta K/\alpha)^{1/\beta}$ we obtain

$$f(t) = \left(\frac{\beta}{a^\alpha \Gamma(\alpha/\beta)} \right) t^{a-1} \exp(-(t/a)^\beta) \tag{15}$$

which is a more customary expression for the density of the generalized gamma density, and the one originally proposed by E. W. Stacy, who first proposed it in 1962

What about stochastic volatility?

- Could this work if we had an equation such as

$$dS_t = \sigma(S_t, \nu_t)dB_t \tag{16}$$

under a risk neutral measure $Q \in \mathbb{Q}$?

- That is, do we have a test such as the one of Delbaen-Shirakawa to tell whether or not S is a martingale or a strict local martingale under $Q \in \mathbb{Q}$?
- The short answer is: No.
- However we can still prove some things; what follows is work with **Jean Jacod**

- First we consider a solution of

$$dX_t = \sigma(X_t)dB_t; \quad X_0 = 1, \tag{17}$$

  that satisfies the Delbaen-Shirakawa conditions so that X is a strict local martingale
- X in (16) is of course also a strong Markov process
- **Theorem:** X in (16) is such that $t \mapsto E(X_t)$ is strictly decreasing
- **Definition:** We let $\mathcal{S}$ denote the class of functions $s : \mathbb{R}^2 \to \mathbb{R}_+$ such that $s(x, v) = 0$ if $x \leq 0$ and that there exists a unique strong solution to (16) (then the solution is necessarily nonnegative, with 0 an absorbing point)

- **Definition:** For $z \geq 0$, we denote by Σ_z the class of all Borel functions σ, vanishing on $(-\infty, 0]$, positive on $(0, \infty)$, satisfying the Delbaen-Shirakawa conditions, and such that for any $t > 0$ and $x, y > z$ we have $p_t^z(x, y) > 0$ for a suitable version of p_t^z.
- **Proposition:** Assuming $s \in \mathcal{S}$, the solution S of (16) is a strict local martingale in the following two situations, where τ is a stopping time and τ' is an $\mathcal{F}_\tau$-measurable variable such that $P(\tau < \tau',\, S_\tau > 0) > 0$:
  (i) The process ν is constant (in time) on the interval $[\tau, \tau')$, and for each v the function $x \mapsto s(x, v)$ belongs to Σ_0.
  (ii) The process ν takes its values in some set Γ on the interval $[\tau, \tau')$, and $s(x, v) = \sigma(x)$ when $y \in \Gamma$, where $\sigma \in \Sigma_0$.

- The main drawback of the previous result is the fact that the time τ' is $\mathcal{F}_\tau$-measurable
- We now relax this assumption, and consider a situation resembling (ii) above with $\Gamma = (\alpha, \infty)$. That is, we still assume $s \in \mathcal{S}$, and also

$$x > \alpha, \; v \in \mathbb{R} \quad \Rightarrow \quad s(x, v) = \sigma(x). \tag{18}$$

- **Theorem:** Assume (18) with a function σ in $\Sigma_\alpha \cap \Sigma_0$ and that the solution S of

$$dS_t = \sigma(S_t, \nu_t) dB_t; \quad S_0 = 1$$

  satisfies $P(\sup_t \nu_t > \alpha) > 0$. Then S is a strict local martingale.

The End

Thank You for Your Attention